Dual Dig Level I (2007) - Solutions

1. Simplify: $\left(\frac{1}{81}\right)^{-3/4}$

Solution:

$$\left(\frac{1}{81}\right)^{-3/4} = (81)^{3/4}$$
$$= \left(\sqrt[4]{81}\right)^3$$
$$= (3)^3$$
$$= 27$$

Answer: 27

2. Find the value of x if $4^{20} + 4^{20} = 2^x$.

Solution:

$$4^{20} + 4^{20} = 2\left(4^{20}\right)$$
$$= 2\left(\left[2\right]^2\right)^{20}$$
$$= 2\left(2^{40}\right)$$
$$= 2^{41}$$

Answer: 41

3. The square shown below can be filled in so that each row and each column contains each of the numbers 1, 2, 3, and 4 exactly once. What does x equal?

4			
		3	
	x		
1			4

Solution:

Red: Can fill in first
Blue italics: Can fill in next
Green bold: Can fill in next
Brown italics bold: Can fill in last

4	*2*	1	3
2	*4*	3	*1*
3	*1*	4	2
1	3	2	4

Answer: 1

4. At a party at which fewer than two-dozen people showed up, two-thirds of the men were married to three-fifths of the women. Assuming all spouses were present, how many people were single?

Solution:

Let m = the number of men at the party.
Let w = the number of women at the party.

If $\frac{2}{3}m = \frac{3}{5}w$, then $m = \frac{9}{10}w$. Since m and, therefore, $\frac{9}{10}w$ must be an integer, and also $m + w < 24$, it must be true that either $w = 10$ or $w = 20$. If $w = 20$, then $m = 18$; however, $20 + 18 = 38$, which is not less than 24. Therefore, $w = 10$ and, thus, $m = 9$, for a total of 19 people. Also, $\left(\frac{3}{5}\right)(10) = 6$ of the women were married, and they were married to 6 of the men. Since 12 of the 19 people were married, 7 must have been single.

Answer: 7 people.

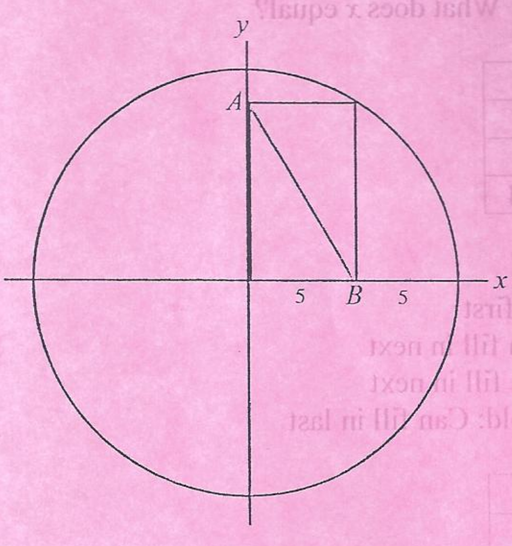
5. Let Ω be a new type of operation where: $a \,\Omega\, b = (a-b)^2 + b$ for any real a and b. Find $(3 \,\Omega\, 2) \,\Omega\, 5$.

Solution:

$$3 \,\Omega\, 2 = (3-2)^2 + 2 = 3 \quad \Rightarrow \quad (3 \,\Omega\, 2) \,\Omega\, 5 = 3 \,\Omega\, 5 = (3-5)^2 + 5 = 9$$

Answer: 9

6. Find the length of $\overline{AB}$ in the figure below.



Solution:

The radius of the circle is 10, so both diagonals of the rectangle have length 10.
Answer: 10.

7. Harry Clotter takes the SAT three times. Harry's second score is 20% lower than his first score. Harry takes Pigwart's test prep program, and his third score is 2000 points, which is 25% higher than his second score. What was Harry's first score?

Longer Solution:

Let x = Harry's first score.
Let y = Harry's second score.

We have:

$$2000 = 1.25y$$
$$2000 = \frac{5}{4}y$$
$$\left(\frac{4}{5}\right)(2000) = y$$
$$y = 1600 \ [\text{points}]$$

Harry's second score is 1600 points, which is 20% lower than his first score, meaning that his second score is 80% of his first score. We have:

$$y = .8x$$
$$1600 = \left(\frac{4}{5}\right)x$$
$$\left(\frac{5}{4}\right)(1600) = x$$
$$x = 2000 \ [\text{points}]$$

Answer: 2000 points.

Shorter Solution:

Let x = Harry's first score.

2000 [points] is 125% of 80% of x.

$$2000 = (1.25)(0.80)x$$
$$2000 = \left(\frac{5}{4}\right)\left(\frac{4}{5}\right)x$$
$$2000 = x$$
$$x = 2000 \ [\text{points}]$$

Answer: 2000 points.

8. Simplify: $\sqrt[3]{\dfrac{125^N \cdot 5^{4N}}{25^{-N}}}$

Solution:

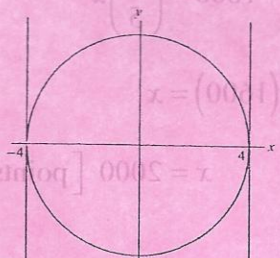
$$\sqrt[3]{\frac{125^N \cdot 5^{4N}}{25^{-N}}} = \sqrt[3]{\frac{\left(5^3\right)^N \cdot 5^{4N}}{\left(5^2\right)^{-N}}}$$
$$= \sqrt[3]{\frac{5^{3N} \cdot 5^{4N}}{5^{-2N}}}$$
$$= \sqrt[3]{5^{3N+4N-(-2N)}}$$
$$= \sqrt[3]{5^{9N}}$$
$$= 5^{3N} \text{ or } 125^N$$

Answer: 5^{3N} or 125^N

9. Find the number of distinct (i.e., different) points at which the graphs of $x^2 + y^2 = 16$ and $x^2 = 16$ intersect in the standard xy-plane.

Solution:

The graph of $x^2 + y^2 = 16$ is the circle below. $x^2 = 16$ is equivalent to $x = \pm 4$, corresponding to the two vertical lines below.



Answer: 2.

10. Simplify: $\dfrac{a^{-2} + b^{-1}}{a^{-1} + b^{-1}}$

Solution:

$$\frac{a^{-2} + b^{-1}}{a^{-1} + b^{-1}} = \frac{\left(\frac{1}{a^2} + \frac{1}{b}\right)}{\left(\frac{1}{a} + \frac{1}{b}\right)} \cdot \frac{a^2 b}{a^2 b}$$
$$= \frac{b + a^2}{ab + a^2} \text{ or } \frac{b + a^2}{a(b + a)}$$

Answer: $\dfrac{b + a^2}{ab + a^2}$ or $\dfrac{b + a^2}{a(b + a)}$

We should remind ourselves that we require $b \neq 0$; it is evident that we require $a \neq 0$ and $b + a \neq 0$.

11. A deck of cards consists of 13 hearts, 13 diamonds, 13 clubs, and 13 spades. (Assume there are no jokers or other extra cards.) At least how many cards must be drawn from the deck in order to be guaranteed that at least one of the drawn cards is a diamond?

Solution:

The 39 non-diamond cards could be drawn first, in which case the 40th card must be a diamond.
Answer: At least 40 cards.

12. Two cars left an intersection at the same time, one heading due north, and the other due west. Some time later, they were exactly 100 miles apart. The car headed north had gone 20 miles farther than the car headed west. How far had each car traveled?

Solution:

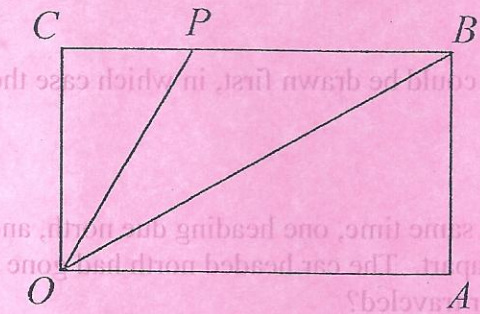
Let x = distance traveled of car headed west.
Let $x+20$ = distance traveled of car headed north.

We can form a right triangle model and use the Pythagorean Theorem.

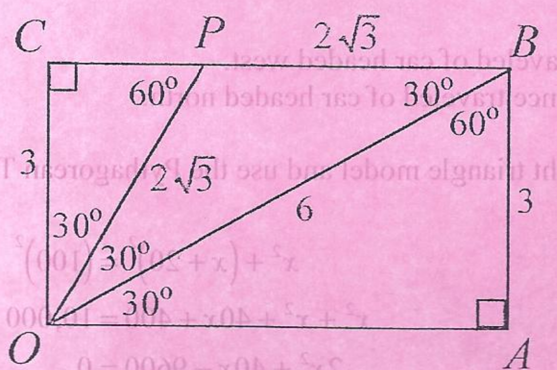
$$x^2+(x+20)^2=(100)^2$$
$$x^2+x^2+40x+400=10{,}000$$
$$2x^2+40x-9600=0$$
$$2(x^2+20x-4800)=0$$
$$2(x+80)(x-60)=0$$

We ignore $x=-80$ as nonsensical and take $x=60$.
Answer: Northbound car traveled 80 miles; westbound car traveled 60 miles.

13. In the rectangle $OABC$ below, side $\overline{OC}$ has length 3. Angles AOB, BOP, and POC each have measure 30°. What is the perimeter of triangle BOP?

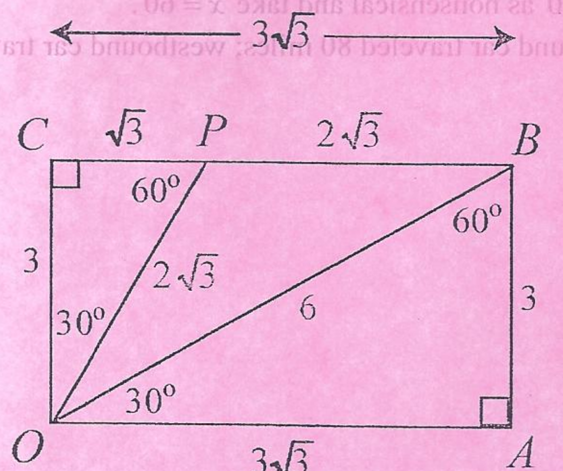


Solution 1:



Triangle BOP is an isosceles triangle; within, congruent angles face congruent sides.
The perimeter of triangle BOP is given by: $2\sqrt{3}+2\sqrt{3}+6=4\sqrt{3}+6$

Solution 2:



The perimeter of triangle BOP is given by: $2\sqrt{3}+2\sqrt{3}+6=4\sqrt{3}+6$

14. Simplify: $\dfrac{1}{\log_4 6}+\dfrac{1}{\log_9 6}$

Solution:

$$\begin{aligned}\frac{1}{\log_4 6}+\frac{1}{\log_9 6}&=\frac{1}{\frac{\ln 6}{\ln 4}}+\frac{1}{\frac{\ln 6}{\ln 9}}\\&=\frac{\ln 4}{\ln 6}+\frac{\ln 9}{\ln 6}\\&=\frac{\ln 4+\ln 9}{\ln 6}\\&=\frac{\ln 36}{\ln 6}\\&=\log_6 36\\&=2\end{aligned}$$

Answer: 2

15. How many lines with equations of the form $y=mx+b$ pass through the point $(3,0)$ in the usual xy-plane if m and b are only allowed to be integers such that $mb>0$?

Solution:

Observe that the point $(3,0)$ lies on the positive x-axis. Also, $mb>0$ implies that m and b have the same sign.

If m and b are positive, then the line cuts through Quadrants I, II, and III, but it cannot intersect the positive x-axis.

If m and b are negative, then the line cuts through Quadrants II, III, and IV, but it cannot intersect the positive x-axis.

Answer: 0

16. Simplify: $\sqrt{0.0007}\sqrt{70{,}000}$

Solution:

$$\begin{aligned}\sqrt{0.0007}\sqrt{70{,}000}&=\sqrt{7\times10^{-4}}\sqrt{7\times10^{4}}\\&=\sqrt{\left(7\times10^{-4}\right)\left(7\times10^{4}\right)}\\&=\sqrt{(7\times7)\left(10^{-4}\times10^{4}\right)}\\&=\sqrt{\left(7^2\right)\left(10^0\right)}\\&=\sqrt{\left(7^2\right)(1)}\\&=7\end{aligned}$$

Answer: 7

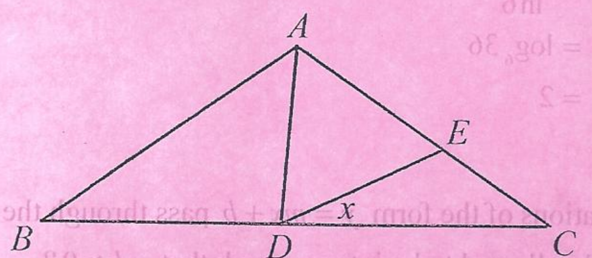
17. 3^{23} ends in what digit?

Solution:

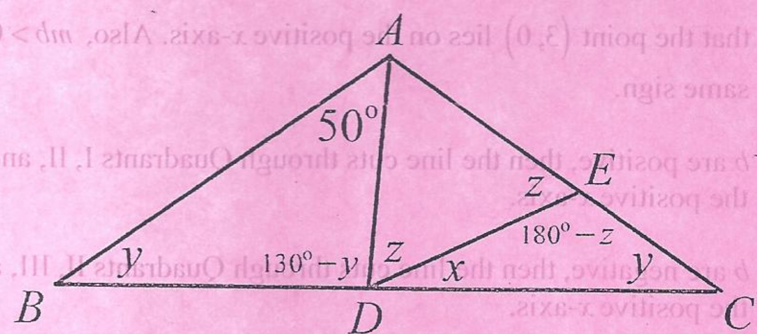
Consider 3^n, where n is a positive integer. The last digits (i.e., the digits in the ones place) of these powers of 3 cycle around 3, 9, 7, and 1, corresponding to the remainders 1, 2, 3, and 0 that are obtained when n is divided by 4. (The powers of i may come to mind.) Consider multiplying the last digit of the previous power of 3 by 3. Observe that: $3\times3=9$, $9\times3=27$, $7\times3=21$, and $1\times3=3$, thus suggesting a cycle.

When we divide 23 by 4, there is a remainder of 3. Therefore, 3^{23} ends in a 7.
Answer: 7

18. In the figure below, $AB=AC$, $AD=AE$, and $\angle BAD=50^\circ$. Find x.



Solution: Let y be the measure of $\angle ABD$.



Because congruent angles face congruent sides in an isosceles or equilateral triangle, the measures of angles ABD and ACD are equal and can be called y in common. Similarly, the measures of angles ADE and AED can be called z in common. Because the sum of the measures of the interior angles of a triangle must be 180°, the measure of angle ADB must be $130^\circ-y$. The angle measure of a straight line segment can be said to be 180°, so we have that $x+z+(130^\circ-y)=180^\circ$, or $x+z-y=50^\circ$. We see from triangle DEC that: $x+y+(180^\circ-z)=180^\circ$, or $x+y=z$. Combining these, we see that:

$$x+(x+y)-y=50^\circ$$
$$2x=50^\circ$$
$$x=25^\circ$$

Answer: 25°

19. Two bathtubs with equal amounts of water begin to be drained at exactly noon today. For each tub, the drainage rate stays constant over time until the tub is emptied, although one tub is drained faster than the other tub. The first tub becomes completely drained at exactly 12:04pm today, while the second tub becomes completely drained at exactly 12:12pm today. In how many minutes after noon does the first tub hold exactly half the amount of water that the second tub holds? Give your answer as an exact fraction or mixed number.

Solution:

Let's call the amount of water in each tub 1 water unit. The first tub is emptied in 4 minutes, so it is being drained at the rate of 1/4 of a water unit per minute. The second tub is emptied in 12 minutes, so it is being drained at the rate of 1/12 of a water unit per minute.

Let t = the number of minutes after noon when the first tub holds half the amount of water that the second tub does.

$$1-\frac{1}{4}t=\frac{1}{2}\left(1-\frac{1}{12}t\right)$$

$$1-\frac{1}{4}t=\frac{1}{2}-\frac{1}{24}t$$

$$24\left(1-\frac{1}{4}t\right)=24\left(\frac{1}{2}-\frac{1}{24}t\right)$$

$$24-6t=12-t$$

$$12=5t$$

$$t=\frac{12}{5}$$

Answer: $\frac{12}{5}$ or $2\frac{2}{5}$ minutes. Note: This corresponds to 12:02:24pm.

20. What is the largest number of pieces into which a circular pie can be cut with 10 straight cuts?

Solution:

We begin with one piece. The nth cut ($n=1,2,3,\ldots,10$) can create a maximum of n new pieces; this is achieved when the cut intersects the $(n-1)$ preexisting cuts. Think: No two cuts should have the same slope. No cut should pass through any intersection point between any two other cuts. Starting with the one whole piece, the first cut creates one new piece, the second cut then creates two new pieces, and so on. The answer is: $1+1+2+3+\ldots+10=56$ pieces.

Answer: 56 pieces.